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Exposing Functionals on $C(Q)$

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1. INTRODUCTION

In this paper, we investigate questions of existence and density of exposing functionals on $C(Q)$, the Banach space of real-valued continuous functions on a compact Hausdorff space Q under the supremum norm. We show that the exposing functionals on $C(Q)$ are dense in the unit sphere of the dual of $C(Q)$ if, and only if, Q is totally disconnected, and admits a strictly positive measure. The above is true if, and only if, the Chebyshev hyperspaces of $C(Q)$ are dense in the set of all hyperspaces of $C(Q)$, provided with the metric θ', introduced by Brown [3]. As a corollary, we show that $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces if μ is σ-finite.

2. NOTATIONS AND TERMINOLOGY

All Banach spaces are over the real field. If X is a Banach Space, X^* stands for the dual space,

$$B(X) = \{x \in X : \|x\| \leqslant 1\} \qquad \text{and} \qquad S(X) = \{x \in X : \|x\| = 1\}.$$

A hyperspace of a Banach space is a closed linear subspace of codimension one. The set of all hyperspaces will be denoted by $\mathbf{H}$. For $M_1, M_2 \in \mathbf{H}$,

$$\theta(M_1, M_2) = \max\left[\operatorname*{Sup}_{m_1 \in S(M_1)} \{ \inf_{m_2 \in M_2} (\|m_1 - m_2\|\}, \operatorname*{Sup}_{m_2 \in S(M_2)} \{ \inf_{m_1 \in M_1} (\|m_1 - m_2\|)\}\right].$$

For $M_1, M_2 \in \mathbf{H}$, $\theta'(M_1, M_2)$ is the Hausdorff distance between $S(M_1)$ and $S(M_2)$. It is easy to see that $\{(M, N) \in \mathbf{H} \times \mathbf{H} : \theta(M, N) < \varepsilon\}_{\varepsilon > 0}$ is a base for a uniformity on $\mathbf{H}$, and the uniform topology on $\mathbf{H}$ is the same as the metric

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topology of $(\mathbf{H}, \theta')$. In fact, we have the following inequalities, the proof of which can be found in [3, 7]: $\frac{1}{2}\theta' \leqslant \theta \leqslant \theta'$.

Let X be a Banach space. An element $f \in S(X^*)$ is said to be an exposing functional on X if there is $x \in S(X)$ such that $f(x) = 1$ and $f(y) < 1$ for every $x \neq y \in B(X)$. In this case, x is said to be an exposed point of $B(X)$ (exposed by f). It is easy to see that every exposed point is an extreme point. Q shall stand for a compact Hausdorff space, and $G(Q)$ for the Banach Space of all real-valued cntinuous functions on Q under the supremum norm.

A measure μ on Q is a regular bounded Borel measure on Q. A measure μ on Q is said to be strictly positive if $\mu(U) > 0$ for every open subset U of Q. Q is said to have property P if Q admits a strictly positive measure. The set of all of compact Hausdorff spaces having property P shall be denoted by $\mathbf{P}$.

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The following theorem establishes a bijection between exposing functionals and Chebyshev hyperspaces.

THEOREM 3.1. *$f \in S(X^*)$ is an exposing functional if, and only if, $M = \operatorname{Ker} f$ is a Chebyshev hyperspace.*

We obtain a relation between the distance between two elements of $S(X^*)$ and the θ'-distance between their kernels.

THEOREM 3.2. *Let $f_1, f_2 \in S(X^*)$, and let $M_i = \operatorname{Ker} f_i$, $i = 1, 2$. Then $\theta'(M_1, M_2) \leqslant 2\, \|f_1 - f_2\|$.*

Proof. Let $m_i \in S(M_1)$. Then

$$\inf_{m_2 \in M_2} (\|m_1 - m_2\|) = d(m_1, M_2) = |f_2(m_1)| = |(f_2 - f_1)(m_1)| \leqslant \|f_2 - f_1\|.$$

Hence,

$$\operatorname*{Sup}_{m_1 \in S(M_1)} \{\inf_{m_2 \in M_2} (\|m_1 - m_2\|)\} \leqslant \|f_2 - f_1\|.$$

Similarly

$$\operatorname*{Sup}_{m_2 \in S(M_2)} \{\inf_{m_1 \in M_1} (\|m_1 - m_2\|)\} \leqslant \|f_2 - f_1\|.$$

Consequently, $\theta(M_1, M_2) \leqslant \|f_2 - f_1\|$, and so $\theta'(M_1, M_2) \leqslant \|f_2 - f_1\|$.

COROLLARY. *If the exposing functionals are dense in $S(X^*)$, then the Chebyshev hyperspaces are dense in $(\mathbf{H}, \theta')$.*

The converse of the above corollary is true. First, we need the following lemma; the proof can be found in [5].

LEMMA. *Let $f, g \in B(X^*)$, and $\varepsilon > 0$. Let $|g(x)| < \varepsilon/2$ whenever $\|x\| \leqslant 1$ and $f(x) = 0$. Then, either $\|f-g\| < \varepsilon$ or $\|f+g\| < \varepsilon$.*

THEOREM 3.3. *Let $f_1, f_2 \in S(X^*)$, let $M_i = \ker f_i$, $i = 1, 2$, and let $\theta'(M_1, M_2) < \varepsilon$. Then, either $\|f-g\| < \varepsilon$, or $\|f+g\| < \varepsilon$.*

Proof. Let $m_1 \in M_1$, $\|m_1\| \leqslant 1$, and $n = m_1/\|m_1\|$. Then

$$|f_2(m_1)| \leqslant |f_2(n)| = d(n, M_2) = \inf_{m_2 \in M_2} (\|n - m_2\|) \leqslant \theta(M_1, M_2)$$
$$\leqslant \theta'(M_1, M_2) < \varepsilon.$$

Hence, by the above lemma, either $\|f_1 - f_2\| < \varepsilon$ or $\|f_1 + f_2\| < \varepsilon$.

COROLLARY. *If the Chebyshev hyperspaces of X are dense in $(\mathbf{H}, \theta')$, then the exposing functionals are dense in $S(X^*)$.*

We apply the above result to $C(Q)$. The dual of $C(Q)$ is $M(Q)$, the Banach space of all regular, bounded Borel measures on Q with the total variation norm.

THEOREM 3.4. *$\mu \in S(M(Q))$ is an exposing measure if, and only if, (i) $S(\mu) = Q$, and*

(ii) $S(\mu^+) \cap S(\mu^-) = \varnothing$,

where $S(\mu)$ stands for the support of μ.

For the proof, refer to [7].

Remarks. (1) if μ is an exposing measure, then μ exposes

$$f \in C(Q),$$

where

$$f(q) = \begin{cases} 1 & \text{if } q \in S(\mu^+) \\ -1 & \text{if } q \in S(\mu^-). \end{cases}$$

(2) If μ is an exposing measure, then $S(|\mu|) = Q$; that is, $|\mu|$ is a strictly positive measure on Q. Thus $Q \in \mathbf{P}$.

The above remark implies, because of Theorem 3.1, that $C(Q)$ has a Chebyshev hyperspace if, and only if, $Q \in \mathbf{P}$. It is therefore important to know when $Q \in \mathbf{P}$.

Separable spaces clearly have property P. On the other hand $\mathbf{P}$ contains many nonseparable spaces since it is closed under products and also contains all compact groups. Some other results on $\mathbf{P}$ are found in [4, 6].

We know that every exposed point is an extreme point. If $Q \in P$, then we have the following theorem.

THEOREM 3.5. *If $Q \in \mathbf{P}$, then every extreme point of $B(C(Q))$ is an exposed point.*

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We now consider conditions on Q which guarantee a dense set of Chebyshev hyperspaces of $C(Q)$—or, equivalently, because of the corollary to Theorem 3.3, conditions on Q which ensures a dense set of exposing functionals in $M(Q)$. We show that this happens if, and only if, $Q \in \mathbf{P}$ and is totally disconnected.

We first need the following theorem.

THEOREM 4.1. *Let X be a Banach space. If the exposing functionals are dense in $S(X^*)$, then the convex hull of the exposed points of $B(X)$ is dense in $B(X)$.*

Proof. Let K be the closed convex hull of the exposed points of $B(X)$. Suppose $K \subsetneq B(X)$. Let $x \in B(X) \backslash K$.

By the Hahn-Banach theorem, there is an $f \in S(X^*)$ and $\alpha > 0$ such that $f(x) > \alpha$ and $f(y) \leqslant \alpha$ for all $y \in K$. Since the exposing functionals are dense in $S(X^*)$, there is an exposing functional of $g \in S(X^*)$ such that $\|f - g\| < 1 - \alpha$. Let g expose $y_0 \in S(X)$. Since $y_0 \in K$, $f(y_0) \leqslant \alpha$. But $(g - f)(y_0) = g(y_0) - f(y_0) = 1 - f(y_0) \geqslant 1 - \alpha$, which implies $\| g - f\| \geqslant 1 - \alpha$, which is a contradiction.

THEOREM 4.2. *The exposing measures are dense in $S(M(Q))$ if, and only if, $Q \in \mathbf{P}$ and Q is totally disconnected.*

Proof. Suppose the exposing measures are dense in $S(M(Q))$. From the remark following Theorem 3.4, $Q \in \mathbf{P}$. By Theorem 4.1, $B(C(Q))$ is the closed convex hull of its extreme points. This implies that Q is totally disconnected [1].

Let us now assume that Q is totally disconnected and $Q \in \mathbf{P}$. By Bishop Phelp's theorem [2], we know that the set of measures in $S(M(Q))$ which attain their norms is dense in $S(M(Q))$. Thus it suffices to show that the set of exposing measures is dense in the set of norm-attaining measures.

Let μ be a norm-attaining measure in $S(M(Q))$, and let $0 < \varepsilon < 1$. By

Phelp's theorem [4], $S(\mu^+) \cap S(\mu^-) = \varnothing$, and μ attains its norm at $f \in s(C(Q))$, where $f = 1$ on $S(\mu^+)$ and $f = -1$ on $S(\mu^-)$.

Since Q is totally disconnected, there is a clopen set V in Q such that $S(\mu^+) \subset V$ and $V \cap S(\mu^-) = \varnothing$. Since $Q \in \mathbf{P}$, there is a strictly positive measure λ on Q with $\lambda(Q) = 1$. Let ν be the Borel measure defined on an arbitrary Borel subset K of Q by

$$\nu(K) = \left(1 - \frac{\varepsilon}{2}\right) \mu(K \cap S(\mu))$$
$$+ \frac{\varepsilon/2}{\lambda(Q \backslash S(\mu))} \left[\lambda(K \cap (V \backslash S(\mu))) - \lambda(K \backslash (S(\mu) \cup V))\right].$$

Then, $\|\nu\| = 1$, $S(\nu^+) = V$, $S(\nu^-) = Q \backslash V$ and $\|\mu - \nu\| < \varepsilon$. Hence, by Theorem 3.3, ν is an exposing measure at a distance less than ε from μ.

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Let (T, Σ, μ) be a σ-finite measure space. $L_\infty(T, \Sigma, \mu)$ is isometrically isomorphic to $C(Q)$ for a totally disconnected compact Hausdorff space. This fact, together with the above results, yields the following theorem.

THEOREM 5.1. *Let (T, Σ, μ) be a σ-finite measure space. Then $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces.*

Proof. Since μ is σ-finite, $L_\infty(\mu)$ is the dual of $L_1(\mu)$. Again, since μ is σ-finite, there is an $f \in L_1(\mu)$ such that $\|f\|_1 = 1$ and $f > 0$ a.e. (μ). This f, viewed as a bounded, linear functional on $L_\infty(\mu)$, is an exposing functional, and exposes the constant function 1.

$L_\infty(\mu)$ is isometrically isomorphic to $C(Q)$ for a totally disconnected, compact Hausdorff space Q. This $C(Q)$ has an exposing functional, and hence $Q \in \mathbf{P}$. Thus, by Theorem 4.2, $C(Q)$, and consequently, $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces.

COROLLARY. *l_∞ has a dense set of Chebyshev hyperspaces.*

The corollary can be seen to be true, also, by observing that $l_\infty = C(\beta N)$, and βN is a totally disconnected, separable, Compact Hausdorff space.

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